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Flows on the Solid Torus Asymptotic to the Boundary*

A. J. SCHWARTZ

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48104

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INTRODUCTION

If X is a topological space and T denotes the real numbers, then by a *flow* we mean a continuous map $\phi : X \times T \to X$ such that $\phi(x, 0) = x$ and $\phi(x, s + t) = \phi(\phi(x, s), t)$. We shall denote $\phi(x, t)$ by x_t. If X is a differentiable manifold and V is a vector field on X, then V is said to generate ϕ where $V_x(f) = d/dt[f(x_t)]_{t=0}$ for every differentiable function, f.

In [4], Seifert raised the question: Does there exist a flow on S^3 which contains no closed, that is, periodic orbit? He showed that if V_o is a vector field on S^3 which generates a flow whose orbits are the fibers of the Hopf fibration and V is sufficiently close to V_o, in the C^o sense, then the flow generated by V must contain at least one closed orbit. Since S^3 is the union of two solid tori whose intersection is a two-dimensional torus, it is of interest to study flows on a solid torus, $K = D^2 \times S^1$, where $D^2 = \{z \mid z$ complex, $|z| \leqslant 1\}$. If it is possible to construct such a flow on K with no closed orbit, then it is possible to construct such a flow on S^3.

Considering K, one might think that if the flow were such that the restriction to the boundary was the irrational flow, then a closed orbit, encircling the "hole" (that is, a closed orbit not contractible to a point) would exist in the interior of K. However, in [1] Fuller has constructed a flow on K whose only closed orbits are null homotopic.

In this paper, we shall approach the problem from a somewhat different standpoint. We consider a flow on K such that every interior orbit approaches the boundary as $t \to \infty$, and show that the boundary of K must contain a closed orbit. Thus if the boundary of K contains no closed orbit, the interior of K is not completely unstable.

We shall not assume that ϕ is generated by a vector field. We will make

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considerable use of the covering of K by $\tilde{K}$ a simply connected, noncompact cylinder.

1. Preliminary Definitions and Propositions

1.1. Definition. Let $p : \tilde{X} \to X$ be a covering of X by $\tilde{X}$ with projection p. If $\tilde{\phi} : \tilde{X} \times T \to \tilde{X}$ and $\phi : X \times T \to X$ are flows such that $p(x_t) = [p(x)]_t$, $\tilde{\phi}$ is said to *cover* ϕ.

As a consequence of the covering homotopy property we have the following proposition:

1.2. Proposition. *If $p : \tilde{X} \to X$ is a covering of X by $\tilde{X}$ and $\phi : X \times T \to X$ is a flow, there exists a unique flow $\tilde{\phi} : \tilde{X} \times T \to \tilde{X}$ which covers ϕ.*

1.3. Notation. D^2 will denote the unit disc, $\{z \mid z \text{ complex}, |z| \leqslant 1\}$, S^1 the unit circle, $\{z \mid |z| = 1\}$, and T^+ the positive real numbers. We also use the notation

$$K = D^2 \times S^1 \quad \text{and} \quad \tilde{K} = D^2 \times T.$$

The covering $p : \tilde{K} \to K$ is defined by $p(d, t) = (d, e^{2\pi i t})$. We denote

$$L = S^1 \times S^1 = \text{boundary of } K$$

and

$$\tilde{L} = S^1 \times T = \text{boundary of } \tilde{K}.$$

We shall also use p to denote the restriction of p to $\tilde{L}$.

We shall consider a flow $\phi : K \times T \to K$ and its covering $\tilde{\phi} : \tilde{K} \times T \to \tilde{K}$. Finally, if (d, t) is in $\tilde{K}$ and l is an integer we denote

$$(d, t) + l = (d, t + l).$$

Since

$$p[(x + l)_t] = [p(x + l)]_t = [p(x)]_t = p(x_t) = p(x_t + l)$$

and $(x + l)_t = x_t + l$ for $t = 0$, it follows from the uniqueness of covering paths that

1.4. $(x + l)_t = (x_t + l)$ for x in $\tilde{K}$, t real, and l an integer.

It follows from the continuity of $\tilde{\phi}$ and the compactness of $\tilde{K}' = D^2 \times [-1, 1]$ that there exists a function $\Delta' : T^+ \times T^+ \to T^+$ such that if

x and y are in $\tilde{K}'$, $\epsilon > 0$, $t > 0$, and $\text{dist}(x, y) < \Delta'(\epsilon, t)$, then $\text{dist}(x_s, y_s) < \epsilon$ for $|s| \leqslant t$. Applying 1.4, we have

1.5. There exists a function $\Delta : T^+ \times T^+ \to T^+$ such that if x and y are in $\tilde{K}$, $\epsilon > 0$, $t > 0$, and $\text{dist}(x, y) < \Delta(\epsilon, t)$, then $\text{dist}(x_s, y_s) < \epsilon$ for $|s| \leqslant t$. (Note: we assume that $K = D^2 \times S^1$ and $\tilde{K} = D^2 \times T$ are equipped with their product metrics.)

We now introduce some concepts from topological dynamics.

1.5. Definition. If x is a point of K or $\tilde{K}$, by the *omega limit set of x*, Ω_x, we mean $\bigcap_{t \in T} \text{cl}\{x_s \mid s \geqslant t\}$; where $\text{cl}\{...\}$ denotes the closure of $\{...\}$.

It is easy to verify that

$$\Omega_x = \{y \in \tilde{K} \mid x_{t(k)} \to y \qquad \text{for some} \qquad t(k) \to \infty\}. \tag{1.6}$$

1.7. Definition. A compact set $M \subset \tilde{K}$ is called *minimal* where M is nonempty invariant (i.e., $M_t = M$ for all t in T), and contains no such proper subset.

By an application of Zorn's lemma one sees that

1.8. Every compact invariant set contains a minimal set.

1.9. Definition. x_T is called a *closed orbit* where x_T is compact. Thus

1.10. x_T is a closed orbit if and only if $x_h = x$ for some $h \neq 0$.

Note that a fixed point, $x_T = \{x\}$ is a closed orbit.

1.11. Definition. Consider a flow in L or $\tilde{L}$. We say a closed orbit is *bounding in L or $\tilde{L}$* if its complement has two components, at least one of which is bounded.

As a consequence of the Brouwer fixed-point theorem we have

1.11. Every bounding orbit in $\tilde{L}$ contains a fixed point in the bounded component of its complement.

Note that by invariance of domain, L or $\tilde{L}$ must be an invariant set in K or $\tilde{K}$.

We shall make use of a somewhat generalized version of the Poincaré-Bendixson theory. As a rule, Poincaré-Bendixson theorems are proved for flows in the plane or the two-dimensional sphere, S^2, generated by continuous vector fields (see, for example, [*2*]). However, the only use made of the vector field is in the construction of transversal line segments. Whitney has shown, [*6*], that transversal line segments may be constructed at any regular point (i.e., nonfixed point) of a two-dimensional flow. Thus we have

1.12. THEOREM (*Poincaré-Bendixson*). *Given a flow on* S^2, *and a point* x *in* S^2 *then*

(a) $\Omega_x = x_T$ *if* x_T *is closed. On the other hand, if* x_T *is not closed we have*

(b) $\Omega_x = \gamma =$ *boundary of* C, *where* C *is an open two cell containing* x *and a fixed point. Moreover, if* S^2 *contains finitely fixed points* $c_1, \ldots, c_n$ *then*

(c) Ω_x *is either a closed orbit or* Ω_x *is the union of some of the fixed points and orbits* $y_T^1, \ldots, y_T^m$ *satisfying* $y_t^j \to c_k$ *as* $t \to -\infty$ *and* $y_t^j \to c_l$ *as* $t \to +\infty$, *for some* k *and* l.

Now, consider a flow on $\tilde{L}$ without fixed points. We may embed $\tilde{L}$ in S^2 so that $S^2 - \tilde{L} = \{(0, 0, -1), (0, 0, +1)\}$. We state that $(0, 0, -1)$ and $(0, 0, +1)$ are fixed points and thereby extend the flow to S^2. If x is in $\tilde{L}$ and Ω_x contains no fixed point, then Ω_x is a closed orbit, nonbounding in $\tilde{L}$. On the other hand, if Ω_x contains one fixed point, $(0, 0, 1)$, $S^2 - \Omega_x$ must contain the other. If, in addition, Ω_x were to contain a regular orbit y_T, then $y_T \cup \{(0, 0, 1)\}$ would separate S^2 into two regions, each containing a fixed point, which is impossible. Thus we have

1.13. If $\tilde{L}$ contains no fixed point, then every omega limit set is either a nonbounding orbit or empty.

We may reformulate this by introducing the following definitions:

1.14. NOTATION. For (d, t) in $\tilde{K}$, denote $\pi(d, t) = t$.

1.15. DEFINITION. Let x be in K, $p(\tilde{x}) = x$. If $\pi(\tilde{x}_t) \to +\infty$ $(-\infty)$ as $t \to +\infty$ we say $x_t \to \infty(-\infty)$ as well as $\tilde{x}_t \to \infty(-\infty)$. Thus, we have as a corollary to 1.13,

1.16. If $\tilde{L}$ contains no closed orbit, and x is in $\tilde{L}$ then either $x_t \to +\infty$ or $x_t \to -\infty$.

It will take a good deal more effort to show that all orbits tend, in some sense uniformly, to the same limit.

2. THE BEHAVIOR OF THE FLOW ON L

In [5], Siegel showed that if L contains no compact orbit, it must contain a cross-section, Γ, that is, a simple closed curve, nowhere tangent to the field generating ϕ, which intersects every orbit.

If Γ were covered by a closed curve $\tilde{\Gamma}$ in $\tilde{L}$. It would be easy to show that every orbit tended to $+\infty$ or every orbit tended to $-\infty$. Although we may construct a covering $p^* : L^* \to L$ so that Γ is covered by a closed curve Γ^*, it may not be possible to extend p^* to a covering of K.

The difficulty to be avoided is exemplified by the following system in the plane:

$$\frac{dx}{dt} = \cos 2\pi y,$$

$$\frac{dy}{dt} = \sin 2\pi y.$$

Here all orbits tend to $-\infty$ except $y = 2\pi k$, $k = 0, +1, +2, \ldots$, which tend to $+\infty$. The orbits $y = 2\pi k$ serve as examples of the following concept:

2.1. Definition. x_T is called a *separatrix* where $x_t \to +\infty(-\infty)$ and there exists $y(k) \to x$ such that for each k, $y(k)_t \to -\infty(+\infty)$.

Our immediate aim is to show that separatrices in L are closed orbits.

As a consequence of 1.13 we have the following lemma:

2.2. Lemma. *If $\tilde{L}$ contains at least one orbit $\tilde{x}_T$, such that $\tilde{x}_t \to \infty$ or $\tilde{x}_t \to -\infty$, but no fixed point, then $\tilde{L}$ contains no closed orbit.*

Proof. If $\tilde{L}$ contains a closed orbit, γ, it must be nonbounding. If we embed $\tilde{L}$ in S^2 as before, γ separates $(0, 0, 1)$ and $(0, 0, -1)$. We may select $\tilde{x}$ in $p^{-1}(x)$ so that $\tilde{x}$ is in the same component of $S^2 - \gamma$ as $(0, 0, -1)$. Thus $\tilde{x}_t \nrightarrow (0, 0, 1)$, which is to say, $x_t \nrightarrow \infty$. Similarly $x_t \nrightarrow -\infty$, which proves the lemma.

Our next lemma limits the amount of time an orbit may remain in a compact portion of $\tilde{L}$.

2.3. Lemma. *If $\tilde{L}$ contains no closed orbit, there exists a function $M : T^+ \to T^+$ such that* $\operatorname{diam}(y_{[0,M(t)]}) > t$ *for all y in $\tilde{L}$ and $t > 0$.*

Proof. Suppose, on the contrary, for some $t_o > 0$ and $\{y(k) \mid k = 1, 2, \ldots\} \subset \tilde{L}$ we have $\operatorname{diam}\{y(k)_{[0,k]}\} \leqslant t_o$. According to 1.4 we may assume $\{y(k)\} \subset S^1 \times [0, 1]$ and by choosing a subsequence, if necessary, we may assume $y(k) \to \bar{y}$.

If $\operatorname{diam} \bar{y}_{[0,\infty)} \leqslant 2t_o$ then $\Omega_{\bar{y}}$ must be a closed orbit, contrary to hypothesis.

If $\operatorname{diam} \bar{y}_{[0,\infty)} > 2t_o$, then $\operatorname{dist}(\bar{y}, \bar{y}_h) > t_o$ for some h, $\operatorname{dist}(y(k), y(k)_h) > t_o$ for sufficiently large k, and for $k > h$ we have $\operatorname{diam}(y(k)_{[0,k]}) \geqslant \operatorname{diam}(y(k)_{[0,h]}) > t_o$ contradicting the supposition. The lemma is proved.

The next lemma, in a sense, limits the "speed" of any orbit.

2.4. Lemma. *There exists a positive number F such that for any x in $\tilde{L}$,*

(i) $|\pi(x_s) - \pi(x)| \leqslant 1$ if s in $[0, F]$, and

(ii) $|\pi(x_t) - \pi(x)| \leqslant \dfrac{t}{F} + 1$ for $t \geqslant 0$.

Proof. Let

$$F_x = \inf\{t \geqslant 0 \mid |\pi(x_t) - \pi(x)| = 1\}$$

and

$$F = \inf\{F_x \mid x \text{ in } \tilde{L}\}.$$

Clearly, F satisfies (i). That $F > 0$ follows from the continuity of $\tilde{\phi}$, the compactness of $\tilde{K}' = D^2 \times [-1, 1]$, and 1.4. Now, if N is an integer such that

$$0 \leqslant (N-1)F \leqslant t \leqslant NF,$$

we have

$$\begin{aligned} |\pi(x_t) - \pi(x)| &\leqslant \sum_{k=0}^{N-2} |\pi(x_{kF}) - \pi(x_{(k+1)F})| + |\pi(x_t) - \pi(x_{(N-1)F})| \\ &\leqslant N \leqslant \frac{t}{F} + 1. \end{aligned}$$

We now come to the key theorem of this section.

2.5. Theorem. *If L contains no fixed point, every separatrix is a closed orbit in L.*

Proof. (See Figure 1). Let x_T be a separatrix. Let us say $x_t \to +\infty$. Suppose x_T is not closed in L. Let $x = p(\tilde{x})$. We may assume $\pi(\tilde{x}) = 0$. (See 1.14).

Let

$$t(k) = \sup\{t \mid \pi(\tilde{x}_t) = k\}$$

and

$$y(k) = \tilde{x}_{t(k)} - k. \qquad \text{(See 1.3).}$$

Thus $\{y(k) | \, k = 1, 2,\ldots\}$ is an infinite subset of $\Pi_o = \{\xi \in \tilde{L} \mid \pi(\xi) = 0\} \subset \tilde{L}$. Moreover,

$$\begin{aligned} &y(k)_t > 0 && \text{for} \quad t > 0, \\ &y(k)_t \to +\infty, \\ &p(y(k)_T) = x_T && \text{for all} \quad k. \end{aligned}$$

We choose

$$r(k) = \inf\{t \geqslant 0 \mid \pi(y(k)_t) = 1\}$$

thus

$$0 < \pi(y(k)_t) < 1 \qquad \text{for} \qquad 0 < t < r(k)$$

so that

$$0 < F \leqslant r(k) \leqslant M(1), \tag{2.6}$$

where F is defined in 2.4 and $M(1)$ is defined in 2.3. Note $x_t \to +\infty$ implies $\tilde{L}$ contains no closed orbit according to 2.2.

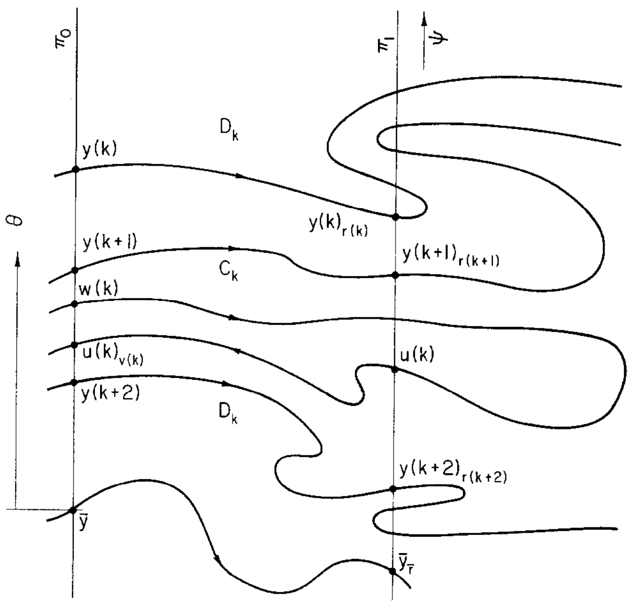

FIG. 1.

Now by taking a subsequence, if necessary, we may assume that $y(k) \to \bar{y}$. Moreover, if we suitably coordinatize a neighborhood U, of Π_0 near $\bar{y}$ by $\theta : U \to T$, we may assume (again taking a subsequence if necessary)

$$\theta(y(1)) > \theta(y(2)) > \cdots > \theta(y(k)) > \cdots.$$

Now, $y(k)_{[0,\infty)} \cup y(k+2)_{[0,\infty)}$ separates $S^1 \times [0, \infty)$ into two components, C_k and D_k, with $y(k+1)$ in C_k.

By supposition, we may assume there is a point $w(k)$ in Π_0 satisfying

(i) $w(k)_t \to -\infty$,

(ii) $w(k)$ is in C_k,

(iii) for some $s > 0$, $\pi(w(k)_s) > 1$, and $w(k)_s$ is in C_k.

We may satisfy (iii) by choosing $s > 0$ such that $\pi(y(k+1)_s) > 1$ and choosing $w(k)$ sufficiently close to $y(k+1)$.

Let

$$s(k) = \sup\{s \mid \pi(w(k)_s) = 1, w(k)_s \in C_k\}$$

and

$$u(k) = w(k)_{s(k)} .$$

Since $u(k)$ is in $w(k)_T$, $u(k)_t \to -\infty$. We may set

$$v(k) = \inf\{v > 0 \mid \pi(u(k)) = 0\}.$$

Now $u(k)_{(0,v(k))}$ cannot cross $y(k)_{[0,\infty)}$ or $y(k+2)_{[0,\infty)}$, thus,

$$u(k)_{(0,v(k))} \subset C_k$$

and

$$\theta(y(k)) > \theta(u(k)_{v(k)}) > \theta(y(k+2)). \tag{2.7}$$

According to 2.4 and 2.3,

$$0 < F \leqslant v(k) \leqslant M(1).$$

Next we suitably coordinatize $V = \Pi_1 - \{y(1)_{r(1)}\}$, where $\Pi_1 = \{\xi \in \tilde{L} \mid \pi(\xi) = 1\}$, by $\psi : V \to T$. We have

$$\psi(y(k)_{r(k)}) > \psi(u(k)) > \psi(y(k+2)_{r(k+2)}) \tag{2.9}$$

so that, according to (2.9) and (2.6),

$$\lim_{k\to\infty} y(k)_{r(k)} = \lim_{k\to\infty} u(k) = \bar{y}_{\bar{r}} ,$$

where

$$\bar{r} = \lim_{k\to\infty} r(k) \geqslant F.$$

(Note that the uniqueness of the limit of $\{y(k)_{r(k)}\}$ implies the uniqueness of the limit of $\{r(k)\}$.) Furthermore, from (2.7) and (2.8) it follows that

$$u(k)_{v(k)} \to \bar{y}_{\bar{r}+\bar{v}} = \bar{y},$$

where

$$\bar{v} = \lim_{k\to\infty} v(k) \geqslant F.$$

Thus $\bar{y}_T$ is a closed orbit in $\tilde{L}$. But, according 2.2, this contradicts the hypothesis. The theorem is proved.

As a corollary to 2.5 we have

2.10. Corollary. *If L contains no closed orbit, all orbits tend to $+\infty$ or all orbits tend to $-\infty$.*

Proof. If L contains no closed orbit, *a fortiori* $\tilde{L}$ contains no closed orbit. Thus, according to 1.16, $\tilde{L} = A \cup B$ where $A = \{x \mid x_t \to \infty\}$ and $B = \{x \mid x_t \to -\infty\}$. But according to 2.5, A and B are closed. Since $A \cap B$ is empty and $\tilde{L}$ is connected, $\tilde{L} = A$ or $\tilde{L} = B$, which was to be shown.

Having established that all orbits tend to the same limit, we now show that they tend to this limit "uniformly".

2.11. THEOREM. *If L contains no closed orbit, there exists a function $\rho : T^+ \to T^+$ such that for any x in $\tilde{L}$,*

$$|\pi(x_s) - \pi(x)| \geqslant t \quad \text{if} \quad x \geqslant \rho(t). \tag{2.12}$$

Proof. Let us assume $x_t \to \infty$ for all x in $\tilde{L}$. For each x in $\tilde{L}$ and $r > 0$, let

$$A_x(r) = \inf\{t \geqslant 0 \mid \pi(x_t) - \pi(x) = r\}$$

and let

$$A(r) = \sup\{A_x(r) | \, x \text{ in } \tilde{L}\}.$$

The finiteness of $A(r)$ follows from the continuity of $\tilde{\phi}$, and 1.4.

Recall that, according to 2.4 (i), if

$$F_x = \inf\{t \geqslant 0 \mid |\pi(x_t) - \pi(x)| = 1\},$$

then

$$F = \inf\{F_x \mid x \text{ in } \tilde{L}\} > 0.$$

Now set $C = A(2 + A(1)/F)$. We assert that

$$\pi(x_t) - \pi(x) \geqslant 1 \quad \text{if} \quad t \geqslant C. \tag{2.13}$$

Suppose $\pi(x_t) - \pi(x) < 1$ for some $t_o \geqslant C$. Then for some u in $[0, C] \subset [0, t_o]$ we have

$$\pi(x_u) - \pi(x) = 2 + \frac{A(1)}{F},$$

and for some $v > t_o$ we have

$$\pi(x_v) - \pi(x) = 1.$$

Now let

$$\bar{u} = \sup \left\{u \text{ in } [0, v] | \, \pi(x_u) - \pi(x) = 2 + \frac{A(1)}{F}\right\}$$

so that

$$\pi(x_s) - \pi(x) < 2 + \frac{A(1)}{F} \quad \text{if} \quad \bar{u} < s \leqslant v, \tag{2.14}$$

and

$$\pi(x_{\bar{u}}) - \pi(x_v) = 1 + \frac{A(1)}{F}.$$

Thus, according to 2.4 (ii), $v - \bar{u} \geqslant A(1)$. But by the definition of A,

$$\pi(x_s) \geqslant \pi(x_{\bar{u}}) + 1 = \pi(x) + \frac{A(1)}{F} + 3$$

for some s in $[\bar{u}, v]$, which contradicts (2.14). Thus (2.13) is proved.

Applying (2.13) we have, for any positive integer N and $t \geqslant NC$,

$$\pi(x_t) - \pi(x) = \pi(x_t) - \pi(x_{(N-1)C}) + \sum_{k=1}^{N-1} \pi(x_{kC}) - \pi(x_{(k-1)C}) \geqslant N.$$

Thus $\rho(t) = (t + 1)\, C$ satisfies (2.12).

Having established that every orbit tends to $+\infty(-\infty)$ uniformly, in the sense of (2.12), *on* L, we turn to a consideration of the flow on K.

3. Proof of the Main Theorem

3.1. Theorem. *Let ϕ be a flow on $K = D^2 \times S^1$ such that $\Omega_x \subset L = S^1 \times S^1 =$ boundary of K for each x in K. Then L contains a closed orbit.*

In order to prove the theorem, we establish a series of lemmas. The first two lemmas extend the conclusions of 2.10 and 2.11 from $\tilde{L}$ to $\tilde{K}$.

3.2. Lemma. *If $\Omega_x \subset L$ for each x in K and L contains no closed orbit, then each orbit in K tends to $+\infty$ or each orbit tends to $-\infty$.*

Proof. Let $\Delta : T^+ \times T^+ \to T^+$ be as in 1.5 and $\delta = \Delta(1/10, \rho(1))$. Since $\Omega_x \subset L$ for each x in K, there exists a function $\sigma : K \to T^+$ such that if x is in K and $s \geqslant \sigma(x)$ then $\operatorname{dist}(x_s, L) < \delta$. Since the metrics on $K = D^2 \times S^1$ and $\tilde{K} = D^2 \times T$ are product metrics we have for x in $\tilde{K}$, and $S(x) = \sigma(p(x))$,

$$\operatorname{dist}(x_s, \tilde{L}) < \delta \qquad \text{for} \qquad s \geqslant S(x).$$

Now for any positive integer n, and x in $\tilde{K}$ we have, setting $S = S(x)$ and $R = \rho(1)$ (where ρ is defined in 2.11),

$$P_n = \pi(x_{S+nR}) - \pi(x_S) = \sum_{k=0}^{n-1} \pi(x_{S+(k+1)R}) - \pi(x_{S+kR}),$$

but

$$Q_k = \pi(x_{S+(k+1)R}) - \pi(x_{S+kR})$$
$$= \pi(x_{S+(k+1)R}) - \pi(y^k_R) + \pi(y^k_R) - \pi(y^k) + \pi(y^k) - \pi(x_{S+kR}),$$

where y^k may be chosen in $\tilde{L}$ so that

$$\text{dist}(y^k, x_{S+kR}) < \delta \leqslant 1/10$$

and thus

$$\text{dist}(y^k_R, x_{S+(k+1)R}) < 1/10.$$

Since y^k is in $\tilde{L}$, $\pi(y^k_R) - \pi(y^k) \geqslant 1$. Therefore $Q_k \geqslant 8/10$ and $P_n \geqslant 8/10n$, and the conclusion of the lemma follows.

In proving 2.11, we used the fact that every orbit in $\tilde{L}$ tended to the same limit, $+\infty$ or $-\infty$, that $\tilde{L}$ was the product of a compact set, S^1, and T, and that the flow on $\tilde{L}$ covered a flow on $L = S^1 \times S^1$. As a factor of $\tilde{L}$, we used no property of S^1 other than its compactness. Therefore the proof of 2.11 may be repeated to obtain

3.3. Lemma. *If L contains no closed orbit, and $\Omega_x \subset L$ for each x in K; then there exists a function, $\omega : T^+ \to T^+$, such that for each x in $\tilde{K}$,*

$$|\pi(x_s) - \pi(x)| \geqslant t \quad \text{if} \quad s \geqslant \omega(t). \tag{3.4}$$

Assuming $x_t \to +\infty$ for each x in $\tilde{K}$, we have the following corollary:

3.5. Corollary. *If L contains no closed orbit and $\Omega_x \subset L$ for each x in K then*

$$\pi(x_{\omega(1)}) - \pi(x) \geqslant 1 \quad \text{and} \quad \pi(x_{-\omega(1)}) - \pi(x) \leqslant -1$$

for each x in K.

Our next aim is to construct a global cross section of $\tilde{K}$. In [*3*], Montgomery and Zippin showed that under certain conditions, a flow in Euclidean space has such a cross section. They employed the existence of local cross sections, as proved by Whitney, [*6*], in their proof. Rather than show that the necessary conditions are present, we shall derive the existence of a global cross section directly. However, the spirit of the derivation owes much to the above-mentioned authors.

We first prove

3.6. Lemma. *There exists a continuous function $W : \tilde{K} \to T$ such that for any x in $\tilde{K}$ and $h > 0$, $W(x_h) - W(x) \geqslant h$.*

Proof. Let $w = \omega(1)$. Define $W : \tilde{K} \to T$ as follows:

$$W(x) = \int_0^w \pi(x_s)\, ds.$$

Then for $h > 0$ we have

$$\begin{aligned} W(x_h) - W(x) &= \int_h^{w+h} \pi(x_s)\, ds - \int_0^w \pi(x_s)\, ds, \\ &= \int_w^{w+h} \pi(x_s)\, ds - \int_0^h \pi(x_s)\, ds, \\ &= \int_0^h (\pi(x_{w+s}) - \pi(x_s))\, ds \geqslant h. \end{aligned}$$

The continuity of W follows from that of $\tilde{\phi}$. The lemma is proved.

3.7. NOTATION. $\sum = \{x \mid W(x) = 0\}$.

3.8. LEMMA. (i) $\sum$ *is bounded in* $\tilde{K}$. (ii) *For each orbit,* x_T, $\sum \cap x_T$ *consists of one point,* $\sigma(x)$. (iii) $\sigma : \tilde{K} \to \sum$ *is continuous.*

Proof. If $R = \max\{| \pi(\xi_t) - \pi(\xi)| \mid | t | \leqslant w\}$ then $| \pi(x)| \leqslant R$ for x in $\sum$ which implies (i).

From (3.4) it follows that for each x, $f(t) = W(x_t)$ is unbounded from above or below. Thus, since f is continuous, each $f(t) = 0$ for some t which implies $\sum \cap x_T$ is not empty. On the other hand, since f is a strictly increasing function, $\sum \cap x_T$ contains but one point. Thus (ii) is valid.

To prove the continuity of σ, we first consider $\psi : \tilde{K} \to T$ defined by $x_{\psi(x)} = \sigma(x)$, which is equivalent to $W(x_{\psi(x)}) = 0$. Now, let $\epsilon > 0$ and $\bar{x}$ in $\tilde{K}$ be given. We then have

$$W(\bar{x}_{[\psi(\bar{x})-\epsilon]}) < 0 < W(\bar{x}_{[\psi(\bar{x})+\epsilon]}).$$

Thus, for y sufficiently close to $\bar{x}$,

$$W(y_{[\psi(\bar{x})-\epsilon]}) < 0 < W(y_{[\psi(\bar{x})+\epsilon]}),$$

so that

$$\psi(\bar{x}) - \epsilon < \psi(y) < \psi(\bar{x}) + \epsilon,$$

which establishes the continuity of ψ. Since $\sigma(x) = x_{\psi(x)}$, σ must be continuous. This completes the proof of 3.8.

3.9. COROLLARY. $\sum$ *has the fixed point property, since* $\sum$ *is a retract of* $\{x \in \tilde{K} \mid -R \leqslant \pi(x) \leqslant R\}$ *under* σ.

We are now ready to complete the proof of 3.1.

Suppose $\Omega_x \subset L$ for each x in K and L contains no closed orbit. Let $\sum$ be as above, let S be an integer greater than $2R + 1$, and let $\sum' = \{x + S \mid x \in \sum\}$. Thus, for any x in $\sum'$ and y in $\sum$ we have

$$\pi(x) > -R + 2R + 1 > \pi(y),$$

so that $\sum \cap \sum'$ is empty. Now let $h : \sum' \to \sum'$ be defined by

$$h(x) = \sigma(x) + S = x_{\psi(x)} + S,$$

where we note that $\psi(x) \neq 0$. According to 3.9, for some $\hat{x}$ in $\sum'$ $h(\hat{x}) = \hat{x}$. Thus, $\hat{x}_{\psi(\hat{x})} = \hat{x} - S$, which implies $[p(\hat{x})]_{\psi(\hat{x})} = p(\hat{x})$, so that $p(\hat{x})_T$ is closed. Since $p(\hat{x})_T = \Omega_{p(\hat{x})} \subset L$, this contradicts our assumption and the theorem is proved.

References

1. Fuller, F. B., Note on trajectories in a solid torus. *Ann. Math.* (2) **56** (1952), 438–439.
2. Hartman, P., "Ordinary Differential Equations." Wiley, New York, 1964, 151–156.
3. Montgomery, D. and Zippin, L., Translation groups of three-space. *Am. J. Math.* **59** (1937), 121–128.
4. Seifert, H., Closed integral curves in 3-space and two-dimensional deformations. *Proc. Am. Math. Soc.* **1** (1950), 287–302.
5. Siegel, C. L., Note on differential equations on the torus. *Ann. Math.* (2) **46** (1945), 423–428.
6. Whitney, H., Regular families of curves. *Ann. Math.* **34** (1933), 240–270.